TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 235, January 1978

RATIONAL APPROXIMATION TO e^{-x}. II

BY

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ABSTRACT. It is shown that as compared to reciprocals of polynomials of degree n, rational functions of degree n provide an effectively better uniform approximation to the function e^{-x} on $[0, \infty)$.

1. Introduction. Let $\mathcal{P}_n$ denote the class of all polynomials of degree at most n, and let

$$\lambda_{m,n} = \inf_{\substack{p_m \in \mathcal{P}_m \\ q_n \in \mathcal{P}_n}} \left\{ \sup_{0 < x < \infty} \left| e^{-x} - \frac{p_m(x)}{q_n(x)} \right| \right\}. \tag{1}$$

It was shown by Cody, Meinardus and Varga [1] that

$$1/6 \leqslant \overline{\lim_{n \to \infty}} (\lambda_{0,n})^{1/n} \leqslant 0.43501 \ldots .$$

Subsequently, Schönhage [6] proved that

$$\lim_{n \to \infty} (\lambda_{0,n})^{1/n} = 1/3. \tag{2}$$

Thus the function e^{-x} can be uniformly approximated on $[0, \infty)$ by reciprocals of polynomials in $\mathcal{P}_n$ with an error going to zero geometrically as n tends to infinity.

As was natural to do, Newman [2] investigated whether it was possible to achieve better than a c^n error by using rational functions of degree n and proved, in effect, that

$$\underline{\lim_{n \to \infty}} (\lambda_{n,n})^{1/n} > 1/1280.$$

Since then we have shown [4] that the number $1/1280$ can be replaced by $1/308$. Although it does not appear to be easy to improve upon the number $1/308$, several people believe that it can be replaced by $1/9$. In conversation with us, Professor P. Erdös wondered why it should be at all possible to approximate e^{-x} on $[0, \infty)$ better by rational functions than by reciprocals of polynomials. Indeed, taking a polynomial $p_m(x)$ other than 1 in (1) does not seem to have any special advantage in approximating a monotonically

Received by the editors June 28, 1976.

AMS (MOS) subject classifications (1970). Primary 41A20, 41A25.

Key words and phrases. Rational approximation, approximation by reciprocals of polynomials, Padé table.

decreasing function like e^{-x} which has no zeros at all. It was remarked by Professor D. J. Newman that at this stage the most interesting thing to do would be to decide whether in approximating by rational functions of degree at most n an error 3^{-n} is all that one could achieve, or is it possible to do better, say even something like 3.0001^{-n}?

We have the answer to this question. We shall indeed prove the following

THEOREM. *There exists* $c > 3$ *such that*

$$\overline{\lim_{n\to\infty}}\,(\lambda_{n,n})^{1/n} \leqslant 1/c. \tag{3}$$

Here c can be taken to be at least 4.09821.

We shall exhibit a rational function giving the desired approximation. It will be constructed by selecting a suitable element from the Padé table of e^x and modifying it in an appropriate way. We shall make essential use of the fact discovered by Padé himself that the element $Q_{j,k}(x)/P_{j,k}(x)$ in the table of e^x, which is given by [3, pp. 246–248]

$$\begin{cases} Q_{j,k}(x) = \displaystyle\sum_{\nu=0}^{k} \frac{\binom{k}{\nu}}{\binom{k+j}{\nu}} \frac{x^\nu}{\nu!}, \\[2ex] P_{j,k}(x) = \displaystyle\sum_{\nu=0}^{j} \frac{\binom{j}{\nu}}{\binom{k+j}{\nu}} \frac{(-x)^\nu}{\nu!}, \end{cases} \tag{4}$$

satisfies

$$P_{j,k}(x) - e^{-x}Q_{j,k}(x) = (-1)^j \frac{x^{j+k+1}}{(j+k)!} \int_0^1 e^{-tx}(1-t)^j t^k\,dt. \tag{5}$$

2. **Lemmas.** In order to work with (5) we shall need the following auxiliary results.

LEMMA 1. *Let r be a positive number and let* $m = [rn]$. *Then uniformly for all* $y \in [0, \infty)$,

$$\begin{aligned} A(r,y) := \lim_{n\to\infty} \left\{ \frac{n^{n+m+1}y^n}{(n+m)!\,Q_{m,n}(ny)} \right\}^{1/n} \\ = \omega^\omega (y(1-\omega))^{1-\omega} \left(\frac{e}{1+r-\omega} \right)^{1+r-\omega}, \end{aligned} \tag{6}$$

where

(7) $$\omega = \left(r + 1 + y - \sqrt{(r+1+y)^2 - 4y}\right)/2.$$

PROOF. It is clear that

$$A(r,y) = \lim_{n\to\infty}\left\{ \frac{yn^{1+(m+1)/n}}{((n+m)!)^{1/n}} \min_{0<\nu<n}\left[\frac{\binom{n}{\nu}(ny)^{\nu}}{\binom{n+m}{\nu}\nu!}\right]^{-1/n}\right\}.$$

According to Stirling's formula we have

$$K! = K^K e^{-K}\sqrt{2\pi K}\, e^{\vartheta/(12K)} \qquad (0 \leqslant \vartheta \leqslant 1).$$

So, setting $\nu = \lambda n$, where $0 \leqslant \lambda \leqslant 1$, we get

$$A(r,y) = y \lim_{n\to\infty} \min_{0<\lambda<1}\left[\frac{\lambda^{\lambda}(1-\lambda)^{1-\lambda}e^{1+m/n-\lambda}}{(1+m/n-\lambda)^{1+m/n-\lambda}y^{\lambda}}\cdot h(\lambda, n)\right]$$

where $h(\lambda, n)$ is independent of y and

$$\lim_{n\to\infty} h(\lambda, n) = 1$$

uniformly in $\lambda \in [0, 1]$. Here 0^0 is to be interpreted as 1. Now it may be verified that

$$\min_{0<\lambda<1} \frac{\lambda^{\lambda}(1-\lambda)^{1-\lambda}e^{1+m/n-\lambda}}{(1+m/n-\lambda)^{1+m/n-\lambda}y^{\lambda}}$$

is attained at the point

$$\omega\left(\frac{m}{n}\right) = \frac{m/n + 1 + y - \sqrt{(m/n+1+y)^2 - 4y}}{2}$$

and, hence,

(8) $$A(r,y) = e^{1+r}\lim_{n\to\infty} f(m/n, y),$$

where

$$f(x,y) = \frac{y}{y^{\omega(x)}}\left(\frac{\omega(x)}{e}\right)^{\omega(x)} \frac{(1-\omega(x))^{1-\omega(x)}}{(1+x-\omega(x))^{1+x-\omega(x)}}.$$

By a straightforward calculation we find that for every fixed $r > 0$ the partial derivative $D_1 f(r, y)$ remains uniformly bounded for $y \in [0, \infty)$. Therefore, the limit in (8) exists uniformly in y and

$$A(r,y) = e^{1+r}f(r,y).$$

LEMMA 2. *Let r be a positive number and let $m = [rn]$. Then in the notation of Lemma* 1,

$$\text{(9)}\qquad \begin{aligned} B(r,y) &:= \lim_{n\to\infty}\left\{|y|^{m+1}\int_0^1\left(e^{-ty}t(1-t)^{m/n}\right)^n dt\right\}^{1/n} \\ &= \left(|y|^r/y\right)e^{-\omega}\omega(1-\omega/y)^r. \end{aligned}$$

The limit exists uniformly in y *for* $y \in [\xi, \infty)$, $\xi \in \mathbf{R}$ *if* $r \in (0, 1)$, *and for* y *belonging to any compact interval if* $r > 1$.

Proof. Putting

$$M(m/n,y) = \max_{0<t<1} e^{-ty}t(1-t)^{m/n}$$

we have

$$B(r,y) = \lim_{n\to\infty}|y^{(m+1)/n}|M(m/n,y)\left\{\int_0^1\left(\frac{e^{-ty}t(1-t)^{m/n}}{M(m/n,y)}\right)^n dt\right\}^{1/n}.$$

As long as y belongs to a compact interval it is easy to see that the limit in (9) exists uniformly in y and

$$B(r,y) = |y^r|M(r,y).$$

On the other hand, by an elementary calculation we obtain, in the notation of Lemma 1,

$$M(r,y) = e^{-\omega}(\omega/y)(1-\omega/y)^r.$$

Thus, for $r \in (0, 1)$ and given $\varepsilon > 0$ there exists a $y^* > 0$ such that

$$|B(r,y)| < y^{(m+1)/n}M(m/n,y) < \varepsilon$$

for all $y \in (y^*, \infty)$. This completes the proof of Lemma 2.

Lemma 3. *Let* r *be a positive number and let* $m = [rn]$. *Then there exists an* $\eta_r > 0$ *such that uniformly for all* $y \in [0, \eta_r]$,

$$\text{(10)}\qquad \begin{aligned} C(r,y) &:= \lim_{n\to\infty}\left(\frac{n^{n+m+1}y^n}{(n+m)!\,Q_{m,n}(-ny)}\right)^{1/n} \\ &\leq \left(\frac{e}{r+1-\alpha}\right)^{r+1-\alpha}\frac{\alpha}{r}\left(1-\frac{\alpha}{r}\right)^{r-\alpha}y^{1-\alpha}e^y, \end{aligned}$$

where

$$\text{(11)}\qquad \alpha = \left(1+r+y-\sqrt{(1+r+y)^2-4ry}\right)/2.$$

Proof. From (5) we get

$$\begin{aligned} &|1/e^{ny} - P_{n,m}(ny)/Q_{n,m}(ny)| \\ &\qquad = \frac{1}{Q_{n,m}(ny)}\,\frac{(ny)^{n+m+1}}{(n+m)!}\int_0^1 e^{-nty}t^m(1-t)^n\,dt. \end{aligned}$$

Therefore; for those values of y for which

$$\frac{1}{e^y} > \lim_{n\to\infty}\left\{\frac{1}{Q_{n,m}(ny)}\frac{(ny)^{n+m+1}}{(n+m)!}\int_0^1 e^{-tny}t^m(1-t)^n\,dt\right\}^{1/n} \tag{12}$$

we must have, in the notation of Lemma 1,

$$\begin{aligned}\lim_{n\to\infty}|P_{n,m}(ny)|^{1/n} &> \lim_{n\to\infty}|Q_{n,m}(ny)|^{1/n}e^{-y}\\ &= \left\{A\left(\frac{1}{r},\frac{y}{r}\right)\right\}^{-r}\left(\frac{e}{r+1}\right)^{r+1} ry^r e^{-y}.\end{aligned} \tag{13}$$

Taking into account the identity

$$P_{n,m}(ny) = Q_{m,n}(-ny)$$

and calculating $A(1/r, y/r)$ with the help of Lemma 1, we obtain (10).

Finally, we use Lemmas 1 and 2 to see that (12) is equivalent to

$$1/e^y > \{A(1/r,y/r)\cdot B(1/r,y/r)\}^r. \tag{14}$$

Since the expression on the right-hand side is $O(y^{1+r})$ as y tends to zero, there exists a finite interval $[0, \eta_r]$ of positive length on which (14) holds. Further, the limit of $|Q_{n,m}(ny)|^{1/n}e^{-y}$ in (13) is uniform for y belonging to this interval.

3. Proof of the Theorem.

Step 1. With the help of Lemmas 1 and 2, we deduce from (5) that for $r \in (0, 1)$ and $m = [rn]$, we have uniformly in $y \in [0, \infty)$,

$$\begin{aligned}F(y) := \lim_{n\to\infty}\left\{\frac{1}{e^{ny}} - \frac{P_{m,n}(ny)}{Q_{m,n}(ny)}\right\}^{1/n} &= A(r,y)\cdot B(r,y)\\ &= \frac{(1-\omega)^{1-\omega}}{(1+r-\omega)^{1+r-\omega}}\left(\frac{\omega}{y}\right)^{1+\omega} e^{1+r-2\omega}y^{1+r}(1-\omega/y)^r.\end{aligned} \tag{15}$$

Here ω and y are connected by the relation

$$y = \omega(1+r-\omega)/(1-\omega). \tag{16}$$

Substituting for y from (16) we obtain

$$F(y) = r^r e^{1+r} G(\omega),$$

where

$$G(\omega) = (1-\omega)^{1-r}\left(\frac{\omega}{1+r-\omega}\right)^{1+r} e^{-2\omega}.$$

As is seen from (16), y increases monotonically from zero to infinity as ω grows from zero to one. Therefore,

$$\sup_{0<y<\infty} F(y) = r^r e^{1+r} \max_{0<\omega<1} G(\omega).$$

By a routine calculation it is verified that $G'(\omega)$ vanishes only if ω is a root of

$$2\omega^3 - (5+r)\omega^2 + 4(1+r)\omega - (1+r)^2$$
$$= (\omega - (1+r)/2)(2\omega^2 - 4\omega + 2(1+r)) = 0,$$

whose only real root is $(1+r)/2$. Now taking into account the uniform convergence in (15) we can conclude that (also see [5])

(17)
$$\lim_{n\to\infty} \sup_{0<x<\infty} \left| \frac{1}{e^x} - \frac{P_{m,n}(x)}{Q_{m,n}(x)} \right|^{1/n}$$
$$= r^r e^{1+r} \max_{0<\omega<1} G(\omega) = r^r \left(\frac{1-r}{2} \right)^{1-r} \geqslant \frac{1}{3}$$

where equality is attained for $r = \frac{1}{3}$ only.

Step 2. We shall now investigate how well $P_{m,n}(x)/Q_{m,n}(x)$ approximates e^{-x} for negative values of x. In view of (5) and Lemmas 1–3 we have uniformly for $y \in [0, \eta_r]$,

(18)
$$\lim_{n\to\infty} \left| \frac{1}{e^{-ny}} - \frac{P_{m,n}(-ny)}{Q_{m,n}(-ny)} \right|^{1/n} = C(r,y) \cdot B(r,-y)$$
$$\leqslant \left(\frac{e}{r+1-\alpha} \right)^{r+1-\alpha} \left(\frac{\alpha}{r} \right)^{\alpha} \left(1 - \frac{\alpha}{r} \right)^{r-\alpha} y^{1-\alpha} e^{y} \cdot y^{r-1} e^{\beta} \beta (1 - \beta/y)^r,$$

where $\beta := y - \alpha$ and α is related to y by

(19)
$$y = \alpha(1+r-\alpha)/(r-\alpha).$$

Further, as α runs from zero to r, y increases from zero to infinity. Thus, substituting for β and y in terms of α, the right-hand side of (18) becomes

(20)
$$e^{r+1+2\alpha/(r-\alpha)} \left(\frac{\alpha}{1+r-\alpha} \right)^{1+r} r^{-r} (r-\alpha)^{r-1}.$$

This quantity does not exceed $r^r((1-r)/2)^{1-r}$ for all $\alpha \in [0, \alpha_r]$, where α_r is an appropriate number in $(0, r)$. Consequently, putting

$$\eta_r' := \alpha_r \frac{1+r-\alpha_r}{r-\alpha_r} \quad \text{and} \quad y_r := \min(\eta_r, \eta_r') > 0,$$

we have uniformly for all $y \in [0, y_r]$,

(21)
$$\lim_{n\to\infty} \left| \frac{1}{e^{-ny}} - \frac{P_{m,n}(-ny)}{Q_{m,n}(-ny)} \right|^{1/n} \leqslant r^r((1-r)/2)^{1-r}.$$

Step 3. (17) and (21) show that for given $\varepsilon > 0$ and all sufficiently large n we have

$$\sup_{-y_r < y < \infty} \left| \frac{1}{e^{ny}} - \frac{P_{m,n}(ny)}{Q_{m,n}(ny)} \right| \leqslant \left(r^r \left(\frac{1-r}{2} \right)^{1-r} + \varepsilon \right)^n;$$

or equivalently,

$$\sup_{0 < x < \infty} \left| \frac{1}{e^x} - e^{-ny_r} \frac{P_{m,n}(x - ny_r)}{Q_{m,n}(x - ny_r)} \right| \tag{22}$$
$$\leqslant \left\{ e^{-y_r} \left(r^r ((1-r)/2)^{1-r} + \varepsilon \right) \right\}^n.$$

At least for $r = \frac{1}{3}$ and sufficiently small $\varepsilon > 0$, the expression within braces attains a value $1/c$ less than $\frac{1}{3}$.

Step 4 (*Calculation of c*). We now investigate quantitatively how well we can do in Step 2. If we use (19) to calculate all the quantities involved in terms of α, we see from (17) and (20) that for $1/c$ we can take the maximum of

$$R_1(r, \alpha) := r^r \left(\frac{1-r}{2} \right)^{1-r} e^{-\alpha(1+r-\alpha)/(r-\alpha)}$$

and

$$R_2(r, \alpha) := \left(\frac{\alpha}{1 + r - \alpha} \right)^{1+r} \frac{(r-\alpha)^{r-1}}{r^r} e^{1+r+\alpha(1-r+\alpha)/(r-\alpha)}$$

provided condition (14) is satisfied. But in terms of α (14) means precisely $R_2(r, \alpha) < 1$. It is, therefore, automatically satisfied as long as the c we are looking for is larger than 1. By a straightforward analysis we find that $R_1(r, \alpha)$ is decreasing whereas $R_2(r, \alpha)$ is increasing as α goes from 0 to r. Thus, for given r the largest value of c that our method allows is $\{R_1(r, \alpha_r)\}^{-1}$, where α_r satisfies

$$R_1(r, \alpha_r) = R_2(r, \alpha_r).$$

Solving this equation by Newton's method we obtain, for $r = 0.4832939$,

$$\alpha_r = 0.099746191233 \ldots$$

and, hence,

$$c = \{R_1(r, \alpha_r)\}^{-1} = 4.09821074569 \ldots$$

as claimed in the Theorem.

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